Dehn surgery on knots in S^3 producing Nil Seifert fibred spaces

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Abstract

We prove that there are exactly 6 Nil Seifert fibred spaces which can be obtained by Dehn surgeries on non-trefoil knots in S^3, with $\{60, 144, 156, 288, 300\}$ as the exact set of all such surgery slopes up to taking the mirror images of the knots. We conjecture that there are exactly 4 specific hyperbolic knots in S^3 which admit Nil Seifert fibred surgery. We also give some more general results and a more general conjecture concerning Seifert fibred surgeries on hyperbolic knots in S^3.

1 Introduction

For a knot K in S^3, we denote by $S^3_K(p/q)$ the manifold obtained by Dehn surgery along K with slope p/q. Here the slope p/q is parameterized by the standard meridian/longitude coordinates of K and we always assume $\gcd(p, q) = 1$. In this paper we study the problem of on which knots in S^3 with which slopes Dehn surgeries can produce Seifert fibred spaces admitting the Nil geometry. Recall that every closed connected orientable Seifert fibred space W admits one of 6 canonical geometries: $S^2 \times \mathbb{R}$, $\mathbb{E}^3$, $\mathbb{H}^2 \times \mathbb{R}$, S^3, Nil, $\widetilde{SL}_2(\mathbb{R})$. More concretely if $e(W)$ denotes the Euler number of W and $\chi(\mathcal{B}_W)$ denotes the orbifold Euler characteristic of the base orbifold $\mathcal{B}_W$ of W, then the geometry of W is uniquely determined by the values of $e(W)$ and $\chi(\mathcal{B}_W)$ according to the following table (cf. §4 of [18]):

	$\chi(\mathcal{B}_W) > 0$	$\chi(\mathcal{B}_W) = 0$	$\chi(\mathcal{B}_W) < 0$
$e(W) = 0$	$\mathbb{S}^2 \times \mathbb{R}$	$\mathbb{E}^3$	$\mathbb{H}^2 \times \mathbb{R}$
$e(W) \neq 0$	$\mathbb{S}^3$	Nil	$\widetilde{SL}_2(\mathbb{R})$

Table 1: The type of geometry of a Seifert fibred space W

Suppose that $S^3_K(p/q)$ is a Seifert fibred space with Euclidean base orbifold. A simple homology consideration shows that the base orbifold of $S^3_K(p/q)$ must be $S^2(2, 3, 6)$ – the 2-sphere with 3 cone points of orders $2, 3, 6$ respectively. The orbifold fundamental group of $S^2(2, 3, 6)$ is the triangle group $\triangle(2, 3, 6) = \langle x, y; x^2 = y^3 = (xy)^6 = 1\rangle$, whose first homology is $\mathbb{Z}/6\mathbb{Z}$. Thus p is divisible by 6. If $p = 0$, then $S^3_K(0)$ must be a torus bundle. By [4], K is a fibred knot with genus one. So K is the trefoil knot or the figure 8 knot. But the 0-surgery on the figure 8 knot is a manifold with the Sol geometry. So K is the trefoil knot, which means that the trefoil knot is the only knot in S^3 and 0 is the only slope which can produce a Seifert fibred space with the Euclidean geometry. Therefore we may assume that $p \neq 0$. Hence $S^3_K(p/q)$ is a Seifert fibred space with the Nil geometry. It is known that on a hyperbolic knot K

in S^3, there is at most one surgery which can possibly produce a Seifert fibred space admitting the Nil geometry and if there is one, the surgery slope is integral [2]. In this paper we show

Theorem 1.1. *Suppose K is a knot in S^3 which is not the (righthanded or lefthanded) trefoil knot $T(\pm 3, 2)$. Suppose that $S^3_K(p/q)$ is a Seifert fibred space admitting the Nil geometry (where we may assume $p, q > 0$ up to changing K to its mirror image). Then $q = 1$ and p is one of the numbers $60, 144, 156, 288, 300$. Moreover we have*
(1) $S^3_K(60) \cong -S^3_{T(3,2)}(60/11)$,
(2) $S^3_K(144) \cong -S^3_{T(3,2)}(144/23)$ or $S^3_K(144) \cong S^3_{T(3,2)}(144/25)$,
(3) $S^3_K(156) \cong S^3_{T(3,2)}(156/25)$,
(4) $S^3_K(288) \cong S^3_{T(3,2)}(288/49)$,
(5) $S^3_K(300) \cong S^3_{T(3,2)}(300/49)$,
where $\cong$ stands for orientation preserving homeomorphism.

Furthermore under the assumptions of Theorem 1.1, we have the following additional information:

Addendum 1.2. (a) The knot K is either a hyperbolic knot or a cable over $T(3, 2)$ as given in Proposition 4.2.

(b) If Case (1) occurs, then K is a hyperbolic knot and its Alexander polynomial is either
$\triangle_K(t) = 1 - t - t^{-1} + t^2 + t^{-2} - t^4 - t^{-4} + t^5 + t^{-5} - t^6 - t^6 + t^7 + t^{-7} - t^8 - t^{-8} + t^9 + t^{-9} - t^{13} - t^{-13} + t^{14} + t^{-14} - t^{15} - t^{-15} + t^{16} + t^{-16} - t^{22} - t^{-22} + t^{23} + t^{-23}$,
or
$\triangle_K(t) = 1 - t^2 - t^{-2} + t^4 + t^{-4} - t^7 - t^{-7} + t^9 + t^{-9} - t^{12} - t^{-12} + t^{13} + t^{-13} - t^{16} - t^{-16} + t^{17} + t^{-17} - t^{21} - t^{-21} + t^{22} + t^{-22}$.

The two Berge knots which yield the lens spaces $L(61, 13)$ and $L(59, 27)$ respectively realize the Nil Seifert surgery with the prescribed two Alexander polynomials respectively. More explicitly these two Berge knots are given in [1], page 6, with $a = 5$ and $b = 4$ in case of Fig. 8, and with $b = 9$ and $a = 2$ in case of Fig. 9, respectively.

(c) If the former subcase of Case (2) occurs, then K is a hyperbolic knot and its Alexander polynomial is
$\triangle_K(t) = 1 - t - t^{-1} + t^2 + t^{-2} - t^4 - t^{-4} + t^5 + t^{-5} - t^6 - t^6 + t^7 + t^{-7} - t^9 - t^{-9} + t^{10} + t^{-10} - t^{11} - t^{-11} + t^{12} + t^{-12} - t^{14} - t^{-14} + t^{15} + t^{-15} - t^{16} - t^{-16} + t^{17} + t^{-17} - t^{19} - t^{-19} + t^{20} + t^{-20} - t^{21} - t^{-21} + t^{22} + t^{-22} - t^{24} - t^{-24} + t^{25} + t^{-25} - t^{26} - t^{-26} + t^{27} + t^{-27} - t^{29} - t^{-29} + t^{30} + t^{-30} - t^{34} - t^{-34} + t^{35} + t^{-35} - t^{39} - t^{-39} + t^{40} + t^{-40} - t^{44} - t^{-44} + t^{45} + t^{-45} - t^{49} - t^{-49} + t^{50} + t^{-50} - t^{54} - t^{-54} + t^{55} + t^{-55}$.
This case is realized on the Eudave-Muñoz knot $k(-2, 1, 6, 0)$ of [3, Propositions 5.3 (1) and 5.4 (2)], which is also a Berge knot on which the 143–surgery yields $L(143, 25)$.

(d) If the latter subcase of Case (2) occurs, then $\triangle_K(t) = \triangle_{T(29,5)}(t)\triangle_{T(3,2)}(t^5)$. If Case (4) or (5) occurs, then $\triangle_K(t) = \triangle_{T(41,7)}(t)\triangle_{T(3,2)}(t^7)$ or $\triangle_K(t) = \triangle_{T(43,7)}(t)\triangle_{T(3,2)}(t^7)$ respectively. All these cases can be realized on certain cables over $T(3, 2)$ as given in Proposition 4.2.

(e) If Case (3) occurs, then either $\triangle_K(t) = \triangle_{T(31,5)}(t)\triangle_{T(3,2)}(t^5)$ or
$\triangle_K(t) = 1 - t^3 - t^{-3} + t^4 + t^{-4} - t^5 - t^{-5} + t^6 + t^{-6} - t^8 - t^{-8} + t^9 + t^{-9} - t^{10} - t^{-10} + t^{11} + t^{-11} - t^{13} - t^{-13} + t^{14} + t^{-14} - t^{15} - t^{-15} + t^{16} + t^{-16} - t^{18} - t^{-18} + t^{19} + t^{-19} - t^{20} - t^{-20} + t^{21} + t^{-21} - t^{23} - t^{-23} + t^{24} + t^{-24} - t^{25} - t^{-25} + t^{26} + t^{-26} - t^{28} - t^{-28} + t^{29} + t^{-29} - t^{30} - t^{-30} + t^{31} + t^{-31} - t^{35} - t^{-35} + t^{36} + t^{-36} - t^{40} - t^{-40} + t^{41} + t^{-41} - t^{45} - t^{-45} + t^{46} + t^{-46} - t^{50} - t^{-50} + t^{51} + t^{-51} - t^{55} - t^{-55} + t^{56} + t^{-56} - t^{60} - t^{-60} + t^{61} + t^{-61}$.
The former subcase can be realized on the $(31, 5)$-cable over $T(3, 2)$, and the latter subcase can be realized on the Eudave-Muñoz knot $k(-3, -1, 7, 0)$, which is also a Berge knot on which the 157–surgery yields $L(157, 25)$.

In other words there are exactly 6 Nil Seifert fibred spaces which can be obtained by Dehn surgeries on non-trefoil knots in S^3 and there are exactly 5 slopes for all such surgeries (while on the trefoil knot $T(3, 2)$, infinitely many Nil Seifert fibred spaces can be obtained by Dehn surgeries, in fact by [11], $S_{T(3,2)}(p/q)$ is a Nil Seifert fibred space if and only if $p = 6q \pm 6$, $p \neq 0$). It seems reasonable to raise the following conjecture.

Conjecture 1.3. *If a hyperbolic knot K in S^3 admits a surgery yielding a Nil Seifert fibred space, then K is one of the four hyperbolic Berge knots given in (b) (c) (e) of Addendum 1.2.*

The method of proof of Theorem 1.1 and Addendum 1.2 follows that given in [10] and [8], where similar results are obtained for Dehn surgeries on knots in S^3 yielding spherical space forms which are not lens spaces or prism manifolds. The main ingredient of the method is the use of the correction terms (also known as the d-invariants) for rational homology spheres together with their spin^c structures, defined in [14]. In fact with the same method we can go a bit further to prove the following theorems.

Theorem 1.4. *For each fixed 2-orbifold $S^2(2,3,r)$ (or $S^2(3,4,r)$), where $r>1$ is an integer satisfying $\sqrt{6r/Q}\notin\mathbb{Z}$ (resp. $\sqrt{12r/Q}\notin\mathbb{Z}$) for each $Q=1,2,...,8$, there are only finitely many slopes with which Dehn surgeries on hyperbolic knots in S^3 can produce Seifert fibred spaces with $S^2(2,3,r)$ (resp. $S^2(3,4,r)$) as the base orbifold.*

Theorem 1.5. *For each fixed torus knot $T(m,n)$, with $m\geq 2$ even, $n>1$, $\gcd(m,n)=1$, and a fixed integer $r>1$ satisfying $\sqrt{mnr/Q}\notin\mathbb{Z}$ for each $Q=1,2...,8$, among all Seifert fibred spaces*

$$\{S^3_{T(m,n)}(\frac{mnq\pm r}{q});\ q>0, \gcd(q,r)=1\}$$

only finitely many of them can be obtained by Dehn surgeries on hyperbolic knots in S^3.

The above results suggest a possible phenomenon about Dehn surgery on hyperbolic knots in S^3 producing Seifert fibred spaces, which we put forward in a form of conjecture.

Conjecture 1.6. *For every fixed 2-orbifold $S^2(k,l,m)$, with all k,l,m larger than 1, there are only finitely many slopes with which Dehn surgeries on hyperbolic knots in S^3 can produce Seifert fibred spaces with $S^2(k,l,m)$ as the base orbifold.*

In the above conjecture we may assume that $\gcd(k,l,m)=1$.

After recall some basic properties of the correction terms in Section 2, we give and prove a more general theorem in Section 3. This theorem together with its proof will be applied in the proofs of Theorem 1.1, Addendum 1.2 and Theorems 1.4 and 1.5, which is the content of Section 4.

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2 Correction terms in Heegaard Floer homology

To any oriented rational homology 3-sphere Y equipped with a Spin^c structure $\mathfrak{s}\in\text{Spin}^c(Y)$, there can be assigned a numerical invariant $d(Y,\mathfrak{s})\in\mathbb{Q}$, called the *correction term* of $(Y,\mathfrak{s})$, which is derived in [14] from Heegaard Floer homology machinery. The correction terms satisfy the following symmetries:

$$d(Y,\mathfrak{s})=d(Y,J\mathfrak{s}),\quad d(-Y,\mathfrak{s})=-d(Y,\mathfrak{s}),\tag{1}$$

where $J\colon \text{Spin}^c(Y)\to\text{Spin}^c(Y)$ is the conjugation.

Suppose that Y is an oriented homology 3-sphere, $K\subset Y$ a knot, let $Y_K(p/q)$ be the oriented manifold obtained by Dehn surgery on Y along K with slope p/q, where the orientation of $Y_K(p/q)$ is induced from that of $Y-K$ which in turn is induced from the given orientation of Y. There is an affine isomorphism $\sigma\colon \mathbb{Z}/p\mathbb{Z}\to\text{Spin}^c(Y_K(p/q))$. See [14, 15] for more details about the isomorphism. We shall identify $\text{Spin}^c(Y_K(p/q))$ with $\mathbb{Z}/p\mathbb{Z}$ via σ but with σ suppressed, writing $(Y_K(p/q),i)$ for $(Y_K(p/q),\sigma(i))$. Note here i is mod (p) defined and sometimes it can appear as an integer larger than or equal to p. The following lemma is contained in [13, 10].

Lemma 2.1. *The conjugation $J: Spin^c(Y_K(p/q))\to Spin^c(Y_K(p/q))$ is given by*

$$J(i)=p+q-1-i,\ \ \textit{for } 0\leq i<p+q.$$

For a positive integer n and an integer k we use $[k]_n \in \mathbb{Z}/n\mathbb{Z}$ to denote the congruence class of k modulo n.

Let $L(p,q)$ be the lens space obtained by p/q–surgery on the unknot in S^3. The correction terms for lens spaces can be computed inductively as in [14]:

$$\begin{aligned} d(S^3,0) &= 0, \\ d(L(p,q),i) &= -\frac{1}{4}+\frac{(2i+1-p-q)^2}{4pq}-d(L(q,[p]_q),[i]_q), \text{ for } 0 \le i < p+q. \end{aligned} \tag{2}$$

For a knot K in S^3, write its Alexander polynomial in the following standard form:

$$\triangle_K(t) = a_0 + \sum_{i \ge 1} a_i(t^i + t^{-i}).$$

For $i \ge 0$, define

$$b_i = \sum_{j=1}^{\infty} j a_{i+j}.$$

Note that the a_i's can be recovered from the b_i's by the following formula

$$a_i = b_{i-1} - 2b_i + b_{i+1}, \text{ for } i > 0. \tag{3}$$

By [15] [16], if $K \subset S^3$ is a knot on which some Dehn surgery produces an L-space, then the b_i's for K satisfy the following properties:

$$b_i \ge 0,\ b_i \ge b_{i+1} \ge b_i - 1,\ b_i = 0 \text{ for } i \ge g(K) \tag{4}$$

and if $S^3_K(p/q)$ is an L-space, where $p,q>0$, then for $0 \le i \le p-1$,

$$d(S^3_K(p/q),i) = d(L(p,q),i) - 2b_{\min\{\lfloor \frac{i}{q} \rfloor, \lfloor \frac{p+q-i-1}{q} \rfloor\}}. \tag{5}$$

This surgery formula has been generalized in [12] to one that applies to any knot in S^3 as follows. Given any knot K in S^3, from the knot Floer chain complex, there is a uniquely defined sequence of integers V_i^{K}, $i \in \mathbb{Z}$, satisfying

$$V_i^{\mathrm{K}} \ge 0, V_i^{\mathrm{K}} \ge V_{i+1}^{\mathrm{K}} \ge V_i^{\mathrm{K}} - 1, \quad V_i^{\mathrm{K}} = 0 \text{ for } i \ge g(K) \tag{6}$$

and the following surgery formula holds

Proposition 2.2. *When $p,q>0$,*

$$d(S^3_K(p/q),i) = d(L(p,q),i) - 2V^K_{\min\{\lfloor \frac{i}{q} \rfloor, \lfloor \frac{p+q-i-1}{q} \rfloor\}}$$

for $0 \le i \le p-1$.

3 Finitely many slopes

Theorems 1.1, 1.4 and 1.5 will follow from the following more general theorem and its proof.

Theorem 3.1. *Let L be a given knot in S^3, and r,l,Q be given positive integers satisfying*

$$\sqrt{\frac{rl}{Q}} \notin \mathbb{Z}. \tag{7}$$

Suppose further that l is even. Then there exist only finitely many positive integers q, such that $S^3_L(\frac{lq \pm r}{q})$ is homeomorphic to $S^3_K(\frac{lq \pm r}{Q})$ for a knot K in S^3.

Remark 3.2. The condition that l is even is not essential. We require this condition to simplify our argument. The condition (7) does not seem to be essential either.

We now proceed to prove Theorem 3.1. Let each of ζ and ε denote an element in $\{1, -1\}$, and let $p = lq + \zeta r$. We may assume that p is positive (as long as $q > r/l$). Assume that

$$S^3_K(\frac{p}{Q}) \cong \varepsilon S^3_L(\frac{p}{q}), \tag{8}$$

where $\epsilon \in \{\pm 1\}$ indicts the orientation and "$\cong$" stands for orientation preserving homeomorphism. Then the two sets

$$\{d(S^3_K(p/Q), i)|\, i \in \mathbb{Z}/p\mathbb{Z}\}, \quad \{d(\varepsilon S^3_L(p/q), i)|\, i \in \mathbb{Z}/p\mathbb{Z}\}$$

are of course equal, but the two parametrizations for Spin^c may not be equal: they could differ by an affine isomorphism of $\mathbb{Z}/p\mathbb{Z}$, that is, there exists an affine isomorphism $\phi\colon \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$, such that

$$d(S^3_K(\frac{p}{Q}), i) = d(\varepsilon S^3_L(\frac{p}{q}), \phi(i)), \quad \text{for } i \in \mathbb{Z}/p\mathbb{Z}.$$

By Lemma 2.1, the fixed point set of the conjugation isomorphism $J\colon \mathrm{Spin}^c(S^3_K(p/Q)) \to \mathrm{Spin}^c(S^3_K(p/Q))$ is

$$\{\frac{Q-1}{2}, \frac{p+Q-1}{2}\} \cap \mathbb{Z}$$

and likewise the fixed point set of $J\colon \mathrm{Spin}^c(\varepsilon S^3_L(p/q)) \to \mathrm{Spin}^c(\varepsilon S^3_L(p/q))$ is

$$\{\frac{q-1}{2}, \frac{p+q-1}{2}\} \cap \mathbb{Z}.$$

As J and ϕ commute, we must have

$$\phi(\{\frac{Q-1}{2}, \frac{p+Q-1}{2}\} \cap \mathbb{Z}) = \{\frac{q-1}{2}, \frac{p+q-1}{2}\} \cap \mathbb{Z}.$$

It follows that the affine isomorphism $\phi : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ is of the form

$$\phi_a(i) = [a(i-b) + \frac{(1-\alpha)p + q - 1}{2}]_p \tag{9}$$

where b is an element of $\{\frac{Q-1}{2}, \frac{p+Q-1}{2}\} \cap \mathbb{Z}$, $\alpha = 0$ or 1, and a is an integer satisfying $0 < a < p$, $\gcd(a, p) = 1$. By (1) and Lemma 2.1, $d(\varepsilon S^3_L(p/q), \phi_a(i)) = d(\varepsilon S^3_L(p/q), \phi_{p-a}(i))$. So we may further assume that

$$0 < a < \frac{p}{2}, \quad \gcd(p, a) = 1. \tag{10}$$

Let

$$\delta^\varepsilon_a(i) = d(L(p, Q), i) - \varepsilon d(S^3_L(p/q), \phi_a(i)). \tag{11}$$

By Proposition 2.2, we have, when $q > r/l$ (so that $p > 0$),

$$\delta^\varepsilon_a(i) = 2V^{\mathrm{K}}_{\min\{\lfloor \frac{i}{Q} \rfloor, \lfloor \frac{p+Q-1-i}{Q} \rfloor\}}. \tag{12}$$

Let $m \in \mathbb{Z}$ satisfy that

$$0 \le a + \frac{(1-\alpha)\zeta r + q - 1}{2} - mq < q,$$

then as $0 < a < p/2$, we have $0 \le m \le \frac{l}{2}$ when $q > 2r$.

Let

$$\kappa(i) = \min\left\{\lfloor \frac{i}{q} \rfloor, \lfloor \frac{p+q-1-i}{q} \rfloor\right\}.$$

Using Proposition 2.2 and (11), we get

$$\begin{aligned} \delta^\varepsilon_a(i) &= d(L(p, Q), i) - \varepsilon d(S^3_L(p/q), \phi_a(i)) \\ &= d(L(p, Q), i) - \varepsilon d(L(p, q), \phi_a(i)) + 2\varepsilon V^{\mathrm{L}}_{\kappa(\phi_a(i))}. \end{aligned} \tag{13}$$

Lemma 3.3. *With the notations and conditions established above, there exists a constant $N = N(r, l, Q, L)$, such that*

$$\left|a - \frac{mp}{l}\right| < N\sqrt{p}$$

for all $q > 2r$.

Proof. It follows from (6) and (12) that

$$\delta_a^\varepsilon(b+1) - \delta_a^\varepsilon(b) = 0 \text{ or } \pm 2. \tag{14}$$

Using (13), (2) and (9), we get

$$\begin{aligned}
& \delta_a^\varepsilon(b+1) - \delta_a^\varepsilon(b) \\
= \quad & \frac{2b+2-p-Q}{pQ} - d(L(Q,[p]_{\mathrm{Q}}),[b+1]_{\mathrm{Q}}) + d(L(Q,[p]_{\mathrm{Q}}),[b]_{\mathrm{Q}}) + 2\varepsilon(V^{\mathrm{L}}_{\kappa(\phi_a(b+1))} - V^{\mathrm{L}}_{\kappa(\phi_a(b))}) \\
& -\varepsilon\big(d(L(p,q), a + \frac{(1-\alpha)p+q-1}{2}) - d(L(p,q), \frac{(1-\alpha)p+q-1}{2})\big).
\end{aligned} \tag{15}$$

When $\zeta = 1$, by the recursive formula (2), we have (note that $a + \frac{(1-\alpha)p+q-1}{2} < p+q$)

$$\begin{aligned}
& d(L(p,q), a + \frac{(1-\alpha)p+q-1}{2}) - d(L(p,q), \frac{(1-\alpha)p+q-1}{2}) \\
= \quad & \frac{(2a-\alpha p)^2 - (\alpha p)^2}{4pq} - d(L(q,r), a - mq + \frac{(1-\alpha)r+q-1}{2}) + d(L(q,r), \frac{(1-\alpha)r+q-1}{2}) \\
= \quad & \frac{a^2 - a\alpha p}{pq} - \frac{(2a-2mq-\alpha r)^2 - (\alpha r)^2}{4qr} \\
& + d(L(r,[q]_r), [a - mq + \frac{(1-\alpha)r+q-1}{2}]_r) - d(L(r,[q]_r), [\frac{(1-\alpha)r+q-1}{2}]_r) \\
= \quad & -\frac{l}{pr}(a - \frac{mp}{l})^2 + \frac{m^2}{l} - m\alpha \\
& + d(L(r,[q]_r), [a - mq + \frac{(1-\alpha)r+q-1}{2}]_r) - d(L(r,[q]_r), [\frac{(1-\alpha)r+q-1}{2}]_r).
\end{aligned}$$

When $\zeta = -1$,

$$\begin{aligned}
& d(L(p,q), a + \frac{(1-\alpha)p+q-1}{2}) - d(L(p,q), \frac{(1-\alpha)p+q-1}{2}) \\
= \quad & \frac{(2a-\alpha p)^2 - (\alpha p)^2}{4pq} - d(L(q,q-r), a - mq + \frac{-(1-\alpha)r+q-1}{2}) + d(L(q,q-r), \frac{-(1-\alpha)r+q-1}{2}) \\
= \quad & \frac{a^2 - a\alpha p}{pq} - \frac{(2a-2mq+\alpha r-q)^2 - (\alpha r-q)^2}{4q(q-r)} \\
& + d(L(q-r,r), a - mq + \frac{-(1-\alpha)r+q-1}{2}) - d(L(q-r,r), \frac{-(1-\alpha)r+q-1}{2}) \\
= \quad & \frac{a^2}{pq} - \frac{\alpha a}{q} - \frac{(a-mq+\alpha r-q)(a-mq)}{q(q-r)} + \frac{(a-mq-(1-\alpha)r)(a-mq)}{(q-r)r} \\
& - d(L(r,[q-r]_r), [a - mq + \frac{-(1-\alpha)r+q-1}{2}]_r) + d(L(r,[q-r]_r), [\frac{-(1-\alpha)r+q-1}{2}]_r) \\
= \quad & \frac{l}{pr}(a - \frac{mp}{l})^2 + \frac{m^2}{l} - m\alpha \\
& - d(L(r,[q]_r), [a - mq + \frac{-(1-\alpha)r+q-1}{2}]_r) + d(L(r,[q]_r), [\frac{-(1-\alpha)r+q-1}{2}]_r).
\end{aligned}$$

Let

$$\begin{aligned} C_0 &= \frac{2b+2-p-Q}{pQ} - d(L(Q,[p]_Q),[b+1]_Q) + d(L(Q,[p]_Q),[b]_Q) + 2\varepsilon(V^{\mathrm{L}}_{\kappa(\phi_a(b+1))} - V^{\mathrm{L}}_{\kappa(\phi_a(b))}) \\ &\quad -\varepsilon\zeta\left(d(L(r,[q]_r),[a-mq+\frac{\zeta(1-\alpha)r+q-1}{2}]_r) - d(L(r,[q]_r),[\frac{\zeta(1-\alpha)r+q-1}{2}]_r)\right), \end{aligned}$$

then the right hand side of (15) becomes

$$\varepsilon\left(\zeta\frac{l}{pr}(a-\frac{mp}{l})^2 - \frac{m^2}{l} + m\alpha\right) + C_0$$

Using (14), we get

$$\frac{l}{pr}(a-\frac{mp}{l})^2 \le 2 + |\frac{m^2}{l} - m\alpha| + |C_0|.$$

Clearly, $|C_0|$ and m are bounded in terms of r, l, Q, L, so the conclusion of the lemma follows. $\diamondsuit$

Lemma 3.4. *Let k be an integer satisfying*

$$0 \le k < \frac{p-(2l+1)r+l}{2Nl^2\sqrt{p}} - \frac{1}{l}. \tag{16}$$

Let

$$i_k = \frac{(1-\alpha)p+q-1}{2} + k(al-mp), \quad j_k = \frac{(1-\alpha)\zeta r+q-1}{2} + k(al-mp).$$

Then

$$\delta_a^\varepsilon(b+lk+1) - \delta_a^\varepsilon(b+lk) = Ak + B + C_k,$$

where

$$\begin{aligned} A &= \varepsilon\zeta\cdot\frac{2(al-mp)^2}{pr} + \frac{2l}{pQ}, \\ B &= \varepsilon\left(\zeta\frac{l}{pr}(a-\frac{mp}{l})^2 - \frac{m^2}{l} + m\alpha\right), \\ C_k &= \frac{2b+2-p-Q}{pQ} - d(L(Q,[p]_Q),[b+lk+1]_Q) + d(L(Q,[p]_Q),[b+lk]_Q) \\ &\quad +2\varepsilon(V^L_{\kappa(\phi_a(b+lk+1))} - V^L_{\kappa(\phi_a(b+lk))}) - \varepsilon\zeta\big(d(L(r,[q]_r),[a-mq+j_k]_r) - d(L(r,[q]_r),[j_k]_r)\big). \end{aligned}$$

Proof. By (16), we have

$$(lk+1)N\sqrt{p} < \frac{p-(2l+1)r+l}{2l} \le \frac{q-2r+1}{2}. \tag{17}$$

It follows from (10), (17) and Lemma 3.3 that

$$0 \le i_k < i_k + a < p+q, \qquad 0 \le j_k, j_k + a - mq < q. \tag{18}$$

For example,

$$\begin{aligned} j_k + a - mq &= j_k + a - m\frac{p-\zeta r}{l} \\ &= \frac{(1-\alpha)\zeta r+q-1}{2} + (lk+1)(a-\frac{mp}{l}) + \frac{m\zeta r}{l} \\ &< \frac{r+q-1}{2} + \frac{q-2r+1}{2} + \frac{r}{2} \\ &= q. \end{aligned}$$

The other inequalities can be verified similarly.

Using (13), we can compute

$$\begin{aligned}
&\delta_a^\varepsilon(b+lk+1)-\delta_a^\varepsilon(b+lk)\\
= \;& \frac{2b+2lk+2-p-Q}{pQ}-d(L(Q,[p]_{\mathrm{Q}}),[b+lk+1]_{\mathrm{Q}})+d(L(Q,[p]_{\mathrm{Q}}),[b+lk]_{\mathrm{Q}})\\
&+2\varepsilon(V^{\mathrm{L}}_{\kappa(\phi_a(b+lk+1))}-V^{\mathrm{L}}_{\kappa(\phi_a(b+lk))})-\varepsilon\big(d(L(p,q),i_k+a)-d(L(p,q),i_k)\big). \qquad (19)
\end{aligned}$$

As in the proof of Lemma 3.3, using (18) and the recursion formula (2), when $\zeta=1$, we can compute

$$\begin{aligned}
&d(L(p,q),i_k+a)-d(L(p,q),i_k)\\
=\;& \frac{(2i_k+2a+1-p-q)^2-(2i_k+1-p-q)^2}{4pq}-d(L(q,r),j_k+a-mq)+d(L(q,r),j_k)\\
=\;& \frac{a(2k(al-mp)+a-\alpha p)}{pq}-\frac{(2j_k+2a-2mq+1-q-r)^2-(2j_k+1-q-r)^2}{4qr}\\
&+d(L(r,[q]_r),[j_k+a-mq]_r)-d(L(r,[q]_r),[j_k]_r)\\
=\;& -\frac{2(al-mp)^2}{pr}k-\frac{l}{pr}(a-\frac{mp}{l})^2+\frac{m^2}{l}-m\alpha+d(L(r,[q]_r),[j_k+a-mq]_r)-d(L(r,[q]_r),[j_k]_r).
\end{aligned}$$

Similarly, when $\zeta=-1$, we get

$$\begin{aligned}
&d(L(p,q),i_k+a)-d(L(p,q),i_k)\\
=\;& \frac{2(al-mp)^2}{pr}k+\frac{l}{pr}(a-\frac{mp}{l})^2+\frac{m^2}{l}-m\alpha-d(L(r,[q]_r),[j_k+a-mq]_r)+d(L(r,[q]_r),[j_k]_r).
\end{aligned}$$

So the right hand side of (19) is $Ak+B+C_k$. $\diamond$

We can now finish the proof of Theorem 3.1. If $S^3_K(p/Q)\cong\varepsilon S^3_L(p/q)$, then (12) holds, so

$$\delta_a^\varepsilon(b+lk+1)-\delta_a^\varepsilon(b+lk)=0 \text{ or } \pm 2 \qquad (20)$$

for all k satisfying (16).

Let A,B,C_k be as in Lemma 3.4. By (7), $A\neq 0$. So $Ak+B+C$ is equal to 0 or ±2 for at most three values of k for any given C. From the expression of C_k, it is evident that there exists a constant integer $M=M(L)$, such that given p,q,a,ε,ζ, as k varies, C_k can take at most MQr values. Thus $Ak+B+C_k$ can be 0 or ±2, i.e. (20) holds, for at most $3MQr$ values of k. But if $p\geq 4l^2N^2(3lMQr+2)^2$, then each of k in $\{0,1,2,\dots,3MQr\}$ satisfies (16) and thus (20) holds for each of these $3MQr+1$ values of k. This contradiction shows that p is bounded above by $4l^2N^2(3lMQr+2)^2$.

4 Seifert surgeries

In this section we prove Theorem 1.1, Addendum 1.2 and Theorems 1.4 and 1.5.

Lemma 4.1. *If W is an oriented Seifert fibred space whose base orbifold is $S^2(2,3,r)$ (or $S^2(3,4,r)$), $r>1$, then W is homeomorphic to some surgery on the torus knot $T(3,2)$ (resp. $T(4,3)$), i.e.*

$$W\cong\varepsilon S^3_{T(3,2)}(\frac{6q+\zeta r}{q}) \quad (\textit{resp. } W\cong\varepsilon S^3_{T(4,3)}(\frac{12q+\zeta r}{q}))$$

for some $\varepsilon,\zeta\in\{1,-1\}$ and some positive integer q.

Proof. The proof is a quick generalization of that of [10, Lemma 3.1]. The Seifert space W has three singular fibres of orders $2,3,r$ (resp. $3,4,r$) respectively. The exterior of the singular fiber of order r in W

is homeomorphic (not necessarily orientation preserving) to the exterior of the torus knot $T(3,2)$ (resp. $T(4,3)$) in S^3 because there is only one Seifert fibred space (up to homeomorphism) with base orbifold $D^2(2,3)$ (resp. $D^2(3,4)$). Now on $T(3,2)$ (resp. $T(4,3)$), a surgery gives Seifert fibred space with base orbifold $S^2(2,3,r)$ (resp. $S^2(3,4,r)$) if and only if the slope is $\frac{6q+\zeta r}{q}$ (resp. $\frac{12q+\zeta r}{q}$), $\gcd(q,r)=1$. We may assume $q>0$ up to change the sign of ζ. $\diamondsuit$

The following proposition classifies satellite knots in S^3 which admit Nil Seifert surgeries.

Proposition 4.2. *Suppose K is a satellite knot and $S^3_K(p/q)$ is a Nil Seifert fibred space with $p/q>0$. Then K is a cable over $T(3,2)$. More precisely, there are four cases for the cable type and the slope:*

cable type	p/q
(29,5)	*144/1*
(31,5)	*156/1*
(41,7)	*288/1*
(43,7)	*300/1*

Proof. Let C be a companion knot of K such that C is itself not a satellite knot. Let V be a solid torus neighborhood of C in S^3 such that K is contained in the interior of V but is not contained in a 3-ball in V and is not isotopic to the core circle of V. Let N be a regular neighborhood of K in V, $M_K=S^3-int(N)$, $M_C=S^3-int(V)$, and let $V_K(p/q)$ be the p/q-surgery of V along K. Then $S^3_K(p/q)=M_K(p/q)=M_C\cup V_K(p/q)$. Since $S^3_K(p/q)$ does not contain incompressible tori, ∂V must be compressible in $S^3_K(p/q)$ and in fact compressible in $V_K(p/q)$. By [5], it follows that either $V_K(p/q)$ has a connected summand W with $0<|H_1(W)|<\infty$, or $V_K(p/q)$ is a solid torus. In the former case, by [17] $V_K(p/q)$ contains a lens space as a connected summand, which contradicts the fact that $S^3_K(p/q)=M_K(p/q)=M_C\cup V_K(p/q)$ is a Nil Seifert fibred space. Hence $V_K(p/q)$ is a solid torus. Now by [5], K is a 0 or 1-bridge braid in V with winding number $w>1$. By [7, Lemma 3.3] the meridian slope of the solid torus $V_K(p/q)$ is p/w^2q and thus $M_K(p/q)=M_C(p/w^2q)$. So C is a torus knot by [2] and then C must be the trefoil knot $T(3,2)$ by [11].

If K is a (s,t)-cable in V (where we may assume $t>1$ is the winding number of K in V), then by [7, Lemma 7.2], $p=stq+\epsilon_1$, $\epsilon_1\in\{\pm1\}$. So $M_K(p/q)=M_C((stq+\epsilon_1)/(t^2q))$. By [11] we should have $stq+\epsilon_1=6t^2q+\epsilon_2 6$, $\epsilon_2\in\{\pm1\}$. So we have $stq-6t^2q=\epsilon_1 5$ or $-\epsilon_1 7$, which implies $q=1$ and $t=5$ or 7.

If $t=5$, then $s=30+\epsilon_1$ and $p=5(30+\epsilon_1)+\epsilon_1$. That is, either K is the $(29,5)$-cable over $T(3,2)$, $q=1$ and $p=144$ or K is the $(31,5)$-cable over $T(3,2)$, $q=1$ and $p=156$. Likewise if $t=7$, K is the $(41,7)$-cable over $T(3,2)$, $q=1$ and $p=288$ or K is the $(43,7)$-cable over $T(3,2)$, $q=1$ and $p=300$.

Now suppose that K is a 1-bridge braid in V. By [6, Lemma 3.2], $q=1$ and $p=\tau w+d$ where w is the winding number of K in V, and τ and d are integers satisfying $0<\tau<w-1$ and $0<d<w$. Hence $M_K(p/q)=M_C(\tau w+d/w^2)$ and by [11] $\tau w+d=6w^2\pm6$. But $6w^2\pm6-\tau w-d\geq 6w^2-6-(w-1)w-w=5w^2-6>0$. We get a contradiction, which means K cannot be a 1-bridge braid in V. $\diamondsuit$

Proof of Theorem 1.1 and Addendum 1.2. Let K be any non-trefoil knot in S^3 such that $S^3_K(p/q)$ is a Nil Seifert space. Up to changing K to its mirror image, we may assume that $p,q>0$. If K is a torus knot, then by [11], no surgery on K can produce a Nil Seifert fibred space. So we may assume that K is not a torus knot. By [2] and Proposition 4.2, $q=1$. We are now going to give a concrete upper bound for p. As noted in Section 1, the base orbifold of $S^3_K(p)$ is $S^2(2,3,6)$. Thus by Lemma 4.1, $S^3_K(p)\cong\varepsilon S^3_{T(3,2)}(p/q)$ with $p=6q+\zeta 6$, for some $\varepsilon,\zeta\in\{1,-1\}$, and $q>0$. As $p\neq 0$, $p/q>1=g(T(3,2))$ which implies that $S^3_K(p)\cong\varepsilon S^3_{T(3,2)}(p/q)$ is an L-space by [15, Corollary 1.4]. Therefore we may use surgery formula (5) instead of Proposition 2.2. Now we apply the proof of Theorem 3.1 (and the notations established there) to our current case with $L=T(3,2)$, $Q=1$, $l=r=6$. Then $m\in\{0,1,2,3\}$, $b\in\{0,p/2\}$, $V_i^{\mathrm{L}}=b_i^{T(3,2)}$

(which is 1 if $i=0$ and 0 if $i>0$), $V_i^{\mathrm{K}} = b_i^{\mathrm{K}}$ and

$$\begin{aligned} C_0 &= \frac{2b+2-p-1}{p} + 2\varepsilon(b^{T(3,2)}_{\kappa(\phi_a(b+1))} - b^{T(3,2)}_{\kappa(\phi_a(b))}) \\ &\quad -\varepsilon\zeta\left(d(L(6,[q]_6),[a-mq+\frac{\zeta(1-\alpha)6+q-1}{2}]_6) - d(L(6,[q]_6),[\frac{\zeta(1-\alpha)6+q-1}{2}]_6)\right), \end{aligned}$$

Using formula (2) one can compute

$$d(L(6,q),i) = \begin{cases} (\frac{5}{4},\frac{5}{12},\frac{-1}{12},\frac{-1}{4},\frac{-1}{12},\frac{5}{12}), & q=1, i=0,1,...,5, \\ (\frac{-5}{12},\frac{1}{12},\frac{1}{4},\frac{1}{12},\frac{-5}{12},\frac{-5}{4}), & q=5, i=0,1,...,5. \end{cases} \tag{21}$$

Thus $|C_0| \le 1+2+\frac{3}{2} < 5$. Since $|\frac{m^2}{6} - m\alpha| \le 2$ for $m=0,1,2,3$ and $\alpha=0,1$, we may take $N=3$. Similarly recall the A, B, and C_k in Lemma 3.4, and in our current case, C_k becomes

$$\begin{aligned} C_k &= \frac{2b+2-p-1}{p} + 2\varepsilon(b^{T(3,2)}_{\kappa(\phi_a(b+lk+1))} - b^{T(3,2)}_{\kappa(\phi_a(b+lk))}) \\ &\quad -\varepsilon\zeta\left(d(L(6,[q]_6),[a-mq+j_k]_6) - d(L(6,[q]_6),[j_k]_6)\right), \end{aligned}$$

which can take at most 18 values as k varies. Thus the bound for p is $4\cdot 6^2\cdot 3^2(3\cdot 6\cdot 18+2)^2$ when $A\neq 0$.

Now we just need to show that in our current case, A is never zero. Suppose otherwise that $A=0$. Then $\varepsilon\zeta=-1$, $(a-\frac{mp}{6})^2=1$, so $a-mq=\zeta m\pm 1$, and by Lemma 3.4

$$\begin{aligned} &\delta_a^\varepsilon(b+lk+1) - \delta_a^\varepsilon(b+lk) = B + C_k \\ &= -\varepsilon\frac{m^2}{6} + \varepsilon m\alpha + (0 \text{ or -1}) + 2\varepsilon(b^{T(3,2)}_{\kappa(\phi_a(b+lk+1))} - b^{T(3,2)}_{\kappa(\phi_a(b+lk))}) \\ &\quad + d(L(6,[q]_6),[\zeta m\pm 1+j_k]_6) - d(L(6,[q]_6),[j_k]_6). \end{aligned}$$

Thus

$$-\varepsilon\frac{m^2}{6} + d(L(6,[q]_6),\zeta m\pm 1+[3\zeta(1-\alpha)+\frac{q-1}{2}]_6) - d(L(6,[q]_6),[3\zeta(1-\alpha)+\frac{q-1}{2}]_6) \tag{22}$$

is integer valued. Using (21), we see that for each of $m=0,1,2,3$, $q\equiv 1,5 \pmod 6$, $\alpha\in\{0,1\}$ and $\zeta\in\{1,-1\}$, the expression given in (22) is never integer valued. This contradiction proves the assertion that $A\neq 0$.

Now for the bounded region of integral slopes for p, one can use computer calculation to locate those possible integral slopes and identify the corresponding Nil Seifert fibred spaces given in Theorem 1.1, applying (5) (2), which yields Theorem 1.1. One can also recover the possible Alexander polynomials for the candidate knots using formula (3). The rest of Addendum 1.2 follows from [11], Proposition 4.2, [3], and direct verification using SnapPy. $\diamondsuit$

Proof of Theorem 1.4. Let K be a hyperbolic knot in S^3 such that $S^3_K(p/Q)$ is a Seifert fibred space whose base orbifold is $S^2(2,3,r)$ (or $S^2(3,4,r)$). By changing K to its mirror image, we may assume that both p and Q are positive integers. By [9] we have $Q\le 8$. So we just need to show that p is bounded above (independent of hyperbolic K).

By Lemma 4.1,

$$S^3_K(p/Q) \cong \varepsilon S^3_{T(3,2)}(\frac{6q+\zeta r}{q}) \ \text{ (resp. } S^3_K(p/Q) \cong \varepsilon S^3_{T(4,3)}(\frac{12q+\zeta r}{q}))$$

for some $\varepsilon,\zeta\in\{1,-1\}$ and some positive integer q. Now applying Theorem 3.1 with $l=6$ and $L=T(3,2)$ (resp. $l=12$ and $L=T(4,3)$), our desired conclusion is true when (7) holds, i.e.

$$\sqrt{\frac{6r}{Q}}\notin\mathbb{Z} \ \text{ (resp. } \sqrt{\frac{12r}{Q}}\notin\mathbb{Z})$$

for each $Q=1,...,8$. $\diamondsuit$

Proof of Theorem 1.5. Let K be a hyperbolic knot in S^3 such that $S^3_K(p/Q) \cong \varepsilon S^3_{T(m,n)}(\frac{mnq+\zeta r}{q})$. Again $Q \leq 8$ and Theorem 3.1 applies with $l = mn$ and $L = T(m,n)$. $\diamondsuit$

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